A MULTIPLICATIVE FORMULA FOR STRUCTURE CONSTANTS IN THE COHOMOLOGY OF FLAG VARIETIES

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ABSTRACT. Let G be a complex semi-simple Lie group and let P, Q be a pair of parabolic subgroups of G such that Q contains P. Consider the flag varieties G/P, G/Q and Q/P. We show that certain structure constants in $H^*(G/P)$ with respect to the Schubert basis can be written as a product of structure constants coming from $H^*(G/Q)$ and $H^*(Q/P)$ in a very natural way. The primary application is to compute Levi-movable structure constants defined by Belkale and Kumar in [2]. We also give a generalization of this product formula in the branching Schubert calculus setting.

1. Introduction

Let G be a connected, simply connected, semisimple complex algebraic group and let $P \subseteq Q$ be a pair of parabolic subgroups. Consider the induced sequence of flag varieties

$$Q/P \hookrightarrow G/P \twoheadrightarrow G/Q. \tag{1}$$

The goal of this paper is to give a simple multiplicative formula connecting the structure coefficients for the cohomology ring of the three flag varieties in (1) with respect to their Schubert bases. Let W be the Weyl group of G and let $W_P \subseteq W_Q \subseteq W$ denote the Weyl groups of P and Q respectively. For any $w \in W^P \simeq W/W_P$, let $\bar{X}_w \subseteq G/P$ denote the corresponding Schubert variety and let $[X_w] \in H^*(G/P) = H^*(G/P, \mathbb{Z})$ denote the Schubert class of $\bar{X}_w$. It is well known that the Schubert classes $\{[X_w]\}_{w\in W^P}$ form an additive basis for cohomology. Similarly, we have Schubert classes $[X_u] \in H^*(G/Q)$ for $u \in W^Q$ and $[X_v] \in H^*(Q/P)$ for $v \in W^P \cap W_Q$. The letters w, u, v will be used to denote Schubert varieties in G/P, G/Q and Q/P respectively. In Lemma 2.1, we show that for any $w \in W^P$, there is a unique decomposition $w = uv$ where $u \in W^Q$ and $v \in W^P \cap W_Q$. Fix $s \geq 2$ and for any $w_1, \ldots, w_s \in W^P$ such that $\sum_{k=1}^s \text{codim } X_{w_k} = \dim G/P$, define the associated structure coefficient (or structure constant) to be the integer c_w where

$$[X_{w_1}] \cdots [X_{w_s}] = c_w[pt] \in H^*(G/P).$$

The following is the main result of this paper:

Theorem 1.1. *Let $w_1, \ldots, w_s \in W^P$ and let $u_k \in W^Q, v_k \in W^P \cap W_Q$ be defined by $w_k = u_k v_k$. Assume that*

$$\sum_{k=1}^{s} \text{codim } X_{w_k} = \dim G/P \quad \textit{and} \quad \sum_{k=1}^{s} \text{codim } X_{u_k} = \dim G/Q. \tag{2}$$

If $c_w, c_u, c_v \in \mathbb{Z}_{\geq 0}$ *are defined by:*

$$\prod_{k=1}^{s}[X_{w_k}] = c_w[pt], \quad \prod_{k=1}^{s}[X_{u_k}] = c_u[pt], \quad \prod_{k=1}^{s}[X_{v_k}] = c_v[pt]$$

in $H^*(G/P), H^*(G/Q), H^*(Q/P)$ *respectively, then* $c_w = c_u \cdot c_v$.

The dimensional conditions in (2) imply that $\sum_{k=1}^{s} \text{codim } X_{v_k} = \dim Q/P$ and hence the associated structure constant c_v is well defined.

To prove Theorem 1.1, we study the geometry of (1). It is well known that if $\prod_{k=1}^{s}[X_{w_k}] = c_w[pt]$, then the number of points in the intersection of generic translates

$$|g_1X_{w_1} \cap \cdots \cap g_sX_{w_s}| = c_w. \tag{3}$$

We show that for a generic choice of $(\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$, the intersection given in (3) projects onto the intersection $\bigcap_{k=1}^{s} g_kX_{u_k} \subseteq G/Q$ with each fiber of the projection containing exactly c_v points. The techniques used in the proof are inspired by Belkale's work in [1].

1.1. **Levi-movability.** The main application of Theorem 1.1 is to show that the product formula applies to "Levi-movable" s-tuples $(w_1, \ldots, w_s) \in W^P$. Let L_P denote the Levi subgroup of P. Belkale and Kumar give the following definition in [2].

Definition 1.2. *The* s*-tuple* $(w_1, \ldots, w_s) \in (W^P)^s$ *is Levi movable or* L_P*-movable if*

$$\sum_{k=1}^{s} \text{codim } X_{w_k} = \dim G/P$$

and for generic $(l_1, \ldots, l_s) \in (L_P)^s$ *the intersection*

$$l_1w_1^{-1}X_{w_1} \cap \cdots \cap l_sw_s^{-1}X_{w_s}$$

is transverse at $eP \in G/P$.

If $(w_1, \ldots, w_s)$ is Levi-movable, then the associated structure constant is not zero. The converse of this statement is generally not true. The following is the second result of this paper.

Theorem 1.3. *Let* $(w_1, \ldots, w_s)$ *be* L_P*-movable and let* $u_k \in W^Q, v_k \in W^P \cap W_Q$ *be defined by* $w_k = u_kv_k$. *The following are true:*

(i) $(u_1, \ldots, u_s)$ *is* L_Q*-movable*
(ii) $(v_1, \ldots, v_s)$ *is* $L_{(L_Q \cap P)}$*-movable.*

As a consequence of Theorem 1.3, if $(w_1, \ldots, w_s)$ is L_P-movable, we can apply the product formula in Theorem 1.1 to its associated structure constant since the conditions in (2) are satisfied. Moreover, since $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_s)$ are also Levi-movable, we can again apply the product formula to decompose their associated structure constants. This reduces the problem of computing structure constants associated to any Levi movable s-tuple to those coming from the cohomology of maximal flag varieties (P is maximal in G). We remark that the author has proved a special case of Theorems 1.1 and 1.3 for type A flag varieties in [11, Theorem 3] and type C flag varieties in his thesis [12].

1.2. **Representation theory and tensor product invariants.** In this section we state a corollary of Theorems 1.1 and 1.3 in regards to representation theory of the group G. Let $X(H)$ denote the character group of a fixed maximal torus $H \subseteq G$ and let $X^+(H)$ denote the set of dominant characters with respect to some fixed Borel subgroup B containing H. For any dominant character $\lambda \in X^+(H)$ of G, let V_λ denote the corresponding irreducible finite dimensional representation of G of highest weight λ. For any $s \geq 2$, define

$$\Gamma(s,G) := \{(\lambda_1,\ldots,\lambda_s) \in X^+(H)^s \otimes_{\mathbb{Z}} \mathbb{Q} \mid \exists\ N > 1,\ (V_{N\lambda_1} \otimes \cdots \otimes V_{N\lambda_s})^G \neq 0\}.$$

The set $\Gamma(s,G)$ forms a convex cone in the vector space $X^+(H)^s \otimes_{\mathbb{Z}} \mathbb{Q}$ and has been studied in the context of Horn's problem on generalized eigenvalues [2, 4, 5]. The set $\Gamma(s,G)$ was initially characterized by Klyachko [7] in type A and later in all types by Berenstein and Sjamaar [3]. These characterizations consist of a list of inequalities parameterized by nonzero products of Schubert classes. In [8], Knutson, Woodard and Tao determined a minimal set of inequalities for type A and most recently, Ressayre [10] determined a minimal set of inequalities in all types. Let Δ denote the set of simple roots of G and let $\Delta(P)$ denote the simple roots associated to the parabolic subgroup $P \subseteq G$. For any $\alpha \in \Delta$, let ω_α denote the corresponding fundamental weight.

Theorem 1.4. *(Ressayre [10]) If $(w_1,\ldots,w^s) \in W^P$ is L_P-movable with associated structure constant $c_w = 1$, then the set of $(\lambda_1,\ldots,\lambda_s) \in \Gamma(s,G)$ such that*

$$\sum_{k=1}^{s} \omega_\alpha(w_k^{-1}\lambda_k) = 0 \quad \forall\ \alpha \in \Delta\backslash\Delta(P)$$

is a face of $\Gamma(s,G)$ whose codimension is of cardinality $|\Delta\backslash\Delta(P)|$. Moreover, any face of $\Gamma(s,G)$ can be described as above, and the list of faces of codimension equal to 1 is irredundant.

Let $F(w_1,\ldots,w_s) \subseteq \Gamma(s,G)$ be the face of $\Gamma(s,G)$ associated to the Levi movable s-tuple $(w_1,\ldots,w_s) \in (W^P)^s$. Necessarily, the associated structure constant $c_w = 1$. Applying Theorems 1.1 and 1.3 yields the following corollary:

Corollary 1.5. *Let $(w_1,\ldots,w_s) \in (W^P)^s$ be L_P-movable with associated structure constant $c_w = 1$ and let $w_k = u_k v_k$ where $u_k \in W^Q$ and $v_k \in W^P \cap W_Q$. Then $F(w_1,\ldots,w_s)$ is an edge of the face $F(u_1,\ldots,u_s)$.*

Proof. By Theorems 1.1 and 1.3, we have that $(u_1,\ldots,u_s)$ is L_Q-movable and that $c_w = c_u \cdot c_v = 1$, where c_w, c_u, c_v are the structure constants associated to $(w_k)_{k=1}^s, (u_k)_{k=1}^s, (v_k)_{k=1}^s$ respectively. Hence $c_u = 1$ and by Theorem 1.4, $F(u_1,\ldots,u_s)$ is a face of $\Gamma(s,G)$ of codimension $|\Delta\backslash\Delta(Q)|$. It suffices to show that if $(\lambda_1,\ldots,\lambda_s) \in F(w_1,\ldots,w_s)$, then $(\lambda_1,\ldots,\lambda_s) \in F(u_1,\ldots,u_s)$. Let $\alpha \in \Delta\backslash\Delta(Q) \subseteq \Delta\backslash\Delta(P)$. Then for any $w \in W^P$ and rational dominant weight λ, we have

$$\omega_\alpha(w^{-1}\lambda) = uv\omega_\alpha(\lambda) = u\omega_\alpha(\lambda) = \omega_\alpha(u^{-1}\lambda)$$

since $v \in W_Q$ acts trivially on any ω_α where $\alpha \in \Delta\backslash\Delta(Q)$. This proves the corollary. $\square$

1.3. **Generalizations to branching Schubert calculus.** In this section, we give generalizations of Theorems 1.1 and 1.3. Let $\tilde{G}$ be any connected semisimple subgroup of G and fix a torus and Borel subgroup $\tilde{H} \subseteq \tilde{B}$ in $\tilde{G}$ such that $\tilde{H} = H \cap \tilde{G}$. Fix $z \in N_G(H)$ such that

$$\tilde{B} = zBz^{-1} \cap \tilde{G}.$$

By [3, Proposition 2.2.6], such z always exist, however may not be unique. For any parabolic subgroup $P \subseteq G$, we define parabolic subgroup $\tilde{P} := zPz^{-1} \cap \tilde{G}$ of $\tilde{G}$. Consider the $\tilde{G}$-equivariant embedding of flag varieties

$$\phi_z : \tilde{G}/\tilde{P} \hookrightarrow G/P$$

defined by $\phi_z(g\tilde{P}) := z^{-1}gzP$. Since z is fixed we will denote ϕ_z by simply ϕ. The problem concerning "branching Schubert calculus" is to compute the pullback

$$\phi^*([X_w]) = \sum_{\tilde{w} \in \tilde{W}^P} c_w^{\tilde{w}}[X_{\tilde{w}}]$$

in terms of the Schubert basis in $H^*(\tilde{G}/\tilde{P})$. If $\dim X_w = \dim G/P - \dim \tilde{G}/\tilde{P}$, then $\phi^*([X_w]) = c_w[pt]$ for some $c_w \in \mathbb{Z}_{\geq 0}$.

Consider the diagonal embedding $\phi = \phi_e : \tilde{G}/\tilde{P} \hookrightarrow (\tilde{G}/\tilde{P})^s$ and let $[X_{w_1} \times \cdots \times X_{w_s}]$ be a Schubert class in $H^*((\tilde{G}/\tilde{P})^s)$. We have that

$$\phi^*([X_{w_1} \times \cdots \times X_{w_s}]) = \prod_{k=1}^{s}[X_{w_k}].$$

Hence the problem of branching Schubert calculus is a generalization of usual Schubert calculus.

Let Q be a parabolic subgroup which contains P and define $\tilde{Q} := zQz^{-1} \cap \tilde{G}$ to be the corresponding parabolic subgroup of $\tilde{G}$. The embedding ϕ induces the maps

$$\phi_1 : \tilde{G}/\tilde{Q} \hookrightarrow G/Q$$
$$\phi_2 : \tilde{Q}/\tilde{P} \hookrightarrow Q/P.$$

given by $\phi_1(g\tilde{Q}) := z^{-1}gzQ$, and $\phi_2 := \phi|_{\tilde{Q}/\tilde{P}}$. The following is an analogue of Theorem 1.1:

Theorem 1.6. *Let $w = uv \in W^P$ where $u \in W^Q$ and $v \in W^P \cap W_Q$. Assume that*

$$\dim X_w = \dim G/P - \dim \tilde{G}/\tilde{P} \quad \text{and} \quad \dim X_u = \dim G/Q - \dim \tilde{G}/\tilde{Q}. \tag{4}$$

If $c_w, c_u, c_v \in \mathbb{Z}_{\geq 0}$ are defined by:

$$\phi^*([X_w]) = c_w[pt], \quad \phi_1^*([X_u]) = c_u[pt], \quad \phi_2^*([X_v]) = c_v[pt]$$

in $H^*(\tilde{G}/\tilde{P}), H^*(\tilde{G}/\tilde{Q}), H^*(\tilde{Q}/\tilde{P})$ respectively, then $c_w = c_u \cdot c_v$.

The techniques used to prove Theorem 1.6 are the same as those used to prove Theorem 1.1, so we only provide a brief overview in Section 5.

As in Section 1.1, we give a special set of $w \in W^P$ which satisfy the assumptions in Theorem 1.6 by generalizing the notion of Levi-movability.

Definition 1.7. *We say $w \in W^P$ is (L_P, ϕ)-movable if for generic $l \in L_P$ the following induced map on tangent spaces is an isomorphism:*

$$\phi_* : T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to \frac{T_{eP}(G/P)}{T_{eP}(lw^{-1}X_w)}.$$

If ϕ is the diagonal embedding, then $w = (w_1, \ldots, w_s)$ is (L_P, ϕ)-movable if and only if w is L_P-movable. We now give an analogue of Theorem 1.3. Let $\tilde{\mathfrak{h}}$ denote the Lie algebra of $\tilde{H}$ and let $\Delta_{\tilde{G}} \subset \tilde{\mathfrak{h}}^*$ denote the simple roots of $\tilde{G}$. Let $\Delta_{\tilde{Q}} \subseteq \Delta_{\tilde{G}}$ denote the set of simple roots corresponding to a parabolic subgroup $\tilde{Q} \subseteq \tilde{G}$. Let $\mathfrak{z}$ denote the Lie algebra of the center of L_Q.

Theorem 1.8. *Assume there exists a vector $\tau \in \tilde{\mathfrak{h}} \cap z\mathfrak{z}z^{-1}$ such that $\alpha(\tau) \geq 0$ for any $\alpha \in \Delta_{\tilde{G}}$ with equality if and only if $\alpha \in \Delta_{\tilde{Q}}$.*
Let $w = uv \in W^P$ where $u \in W^Q$ and $v \in W^P \cap W_Q$. If w is (L_P, ϕ)-movable, then the following are true:

(i) *u is (L_Q, ϕ_1)-movable*
(ii) *v is $(L_{(L_Q \cap P)}, \phi_2)$-movable.*

The existence of $\tau \in \mathfrak{h}$ in Theorem 1.8 is a restriction on the choice of $Q \subseteq G$. In the case of the diagonal embedding, the vector τ exists if and only if the parabolic subgroup $Q \subseteq G = \tilde{G}^s$ is of the form $Q = \tilde{Q}^s$ for some parabolic subgroup $\tilde{Q} \subset \tilde{G}$.

Theorem 1.8 implies that if $w \in W^P$ is (L_P, ϕ)-movable, then w satisfies the conditions in (4). As with Theorem 1.6, the proof of Theorem 1.8 follows the same outline as the proof in the diagonal embedding case.

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2. Preliminaries

Let G be a connected, simply connected, semisimple complex algebraic group. Fix a Borel subgroup B and a maximal torus $H \subseteq B$. Let $W := N_G(H)/H$ denote the Weyl group of G where $N_G(H)$ is the normalizer of H in G. Let $P \subseteq G$ be a standard parabolic subgroup (P contains B) and let L_P denote the Levi subgroup of P. Denote the Lie algebras of G, H, B, P, L_P by the corresponding frankfurt letters $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \mathfrak{p}, \mathfrak{l}_P$.

Let $R \subseteq \mathfrak{h}^*$ be the set of roots and let $R^{\pm} \subseteq R$ denote the set of positive roots (negative roots). Let R_P denote the set of roots corresponding to $\mathfrak{l}_P$ and let $R_P^{\pm}$ denote the set of positive roots (negative roots) with respect to the Borel subgroup $B_P := B \cap L_P$ of L_P.

Let W^P be the set of minimal length representatives of the coset space W/W_P where W_P is the Weyl group of P (or L_P). For any $w \in W^P$, define the Schubert cell

$$X_w := BwP/P \subseteq G/P.$$

We denote the cohomology class of the closure $\bar{X}_w$ by $[X_w] \in H^*(G/P)$. We begin with some basic lemmas on the Weyl group W.

Lemma 2.1. *The map* $\tau : W^Q \times (W^P \cap W_Q) \to W^P$ *given by* $(u, v) \mapsto uv$ *is well defined and a bijection.*

Proof. Since $W = \bigsqcup_{u \in W^Q} uW_Q$, we have that $W/W_P = \bigsqcup_{u \in W^Q} uW_Q/W_P$. It suffices to show that if $v \in W^P \cap W_Q$, then $uv \in W^P$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the length function on W. For any $v' \in W_P$ we have that

$$\ell(uvv') = \ell(u) + \ell(vv') = \ell(u) + \ell(v) + \ell(v') = \ell(uv) + \ell(v')$$

since $u \in W^Q$, $vv' \in W_Q$, $v \in W^P$ and $v' \in W_P$. Hence $uv \in W^P$. □

Lemma 2.1 shows that for any $w \in W^P$, there is a unique $u \in W^Q$ and $v \in W^P \cap W_Q$ such that $w = uv$. We will assume this relationship between w, u, v given any $w \in W^P$. If these groups elements are indexed $w_k \in W^P$, then we write $w_k = u_k v_k$ accordingly.

Note that the flag variety $Q/P \simeq L_Q/(L_Q \cap P)$ where L_Q is the Levi subgroup of Q. Under this identification, the Schubert cell $X_v \simeq B_Q v(L_Q \cap P)/(L_Q \cap P)$ where $B_Q := B \cap L_Q$. To any $w \in W^P$, we associate the set of roots

$$w \to w^{-1}R^+ \cap R^- \backslash R_P^-.$$

This set is determined by the tangent space

$$T_{eP}(w^{-1}X_w) = \bigoplus_{\alpha \in (w^{-1}R^+ \cap R^- \backslash R_P^-)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the root space of α in $\mathfrak{g}$. Similarly, for any $u \in W^Q$ and $v \in W^P \cap W_Q$ we have

$$u \to u^{-1}R^+ \cap R^- \backslash R_Q^-$$
$$v \to v^{-1}R_Q^+ \cap R_Q^- \backslash R_P^-.$$

Lemma 2.2. *Let* $w = uv \in W^P$. *For any* $q \in Q$, *we have that* $qu^{-1}X_w \cap Q/P = qX_v$.

Proof. Since we can translate the intersection by $(qv)^{-1} \in Q$, it suffices to show that the set of roots corresponding to w intersected with the roots of Q/P equal the roots corresponding to v. More precisely, we need to show that the following two sets are same:

$$w^{-1}R^+ \cap R_Q^- \backslash R_P^- \quad \text{and} \quad v^{-1}R_Q^+ \cap R_Q^- \backslash R_P^-.$$

Since $v^{-1}R_Q = R_Q$, we have that

$$\begin{aligned} w^{-1}R^+ \cap R_Q^- \backslash R_P^- &= (w^{-1}R^+ \cap R_Q^- \backslash R_P^-) \cap (v^{-1}R_Q^- \cup v^{-1}R_Q^+) \\ &= v^{-1}(R_Q^- \cap u^{-1}R^+ \cap vR_Q^- \backslash R_P^-) \cup (v^{-1}R_Q^+ \cap w^{-1}R^+ \cap R_Q^- \backslash R_P^-). \end{aligned}$$

By [9, Excerise 1.3.E], we have that $uR_Q^- \subseteq R^-$ and hence

$$R_Q^- \cap u^{-1}R^+ = \emptyset.$$

This reduces the above calculation to

$$v^{-1}R_Q^+ \cap w^{-1}R^+ \cap R_Q^- \backslash R_P^-.$$

Moreover, we have that $v^{-1}R_Q^+ \subseteq w^{-1}R^+$ since $uR_Q^+ \subseteq R^+$. Thus

$$w^{-1}R^+ \cap R_Q^- \backslash R_P^- = v^{-1}R_Q^+ \cap R_Q^- \backslash R_P^-.$$

This proves the lemma. □

3. The main result

In this section we prove Theorem 1.1. Assume we have $(w_1, \dots, w_s) \in (W^P)^s$ which satisfy the conditions (2) and let $w_k = u_k v_k$ with respect to Lemma 2.1. We begin by considering the following G-variety. Define

$$Y = Y(u_1, \dots, u_s) := \{(\bar{g}\,;\, \bar{g}_1, \dots, \bar{g}_s) \in G/Q \times (G/B)^s \mid \bar{g} \in \bigcap_{k=1}^{s} g_k X_{u_k}\}$$

where the action G on Y is the diagonal action. We now prove that Y is smooth and irreducible. Define

$$\tilde{Y} := G \times_Q (Qu_1^{-1}B/B \times \cdots \times Qu_s^{-1}B/B).$$

Lemma 3.1. *The G-equivariant map $\xi : \tilde{Y} \to Y$ given by*

$$\xi((g\,;\, \overline{q_1 u_1^{-1}}, \dots, \overline{q_s u_s^{-1}})) = (\bar{g}\,;\, \overline{g q_1 u_1^{-1}}, \dots, \overline{g q_s u_s^{-1}}). \tag{5}$$

is well defined and an isomorphism. Moreover, Y is smooth and irreducible.

Proof. If ξ is an isomorphism, then the irreducibility and smoothness of Y follows from the irreducibility and smoothness of $\tilde{Y}$. By [2, Lemma 1], for any $u \in W^Q$ if $e \in gu^{-1}X_u$, then $gu^{-1}X_u = qu^{-1}X_u$ for some $q \in Q$. Hence the map ξ is well defined. Let $(\bar{g}\,;\, \bar{g}_1, \dots, \bar{g}_s) \in Y$. For each k we have $g \in g_k B u_k Q$ and hence

$$g_k = g q_k u_k^{-1} b_k$$

for some $q_k \in Q$ and $b_k \in B$. Computing the inverse image of the point $(\bar{g}\,;\, \bar{g}_1, \dots, \bar{g}_s) \in Y$, we have

$$\xi^{-1}((\bar{g}\,;\, \bar{g}_1, \dots, \bar{g}_s)) = \{(gq\,;\, \overline{q^{-1} q_1 u_1^{-1}}, \dots, \overline{q^{-1} q_s u_s^{-1}}) \mid q \in Q\} \subseteq \tilde{Y}.$$

Hence ξ is surjective. Moreover, we have that

$$(gq\,;\, \overline{q^{-1} q_1 u_1^{-1}}, \dots, \overline{q^{-1} q_s u_s^{-1}}) \sim (g\,;\, \overline{q_1 u_1^{-1}}, \dots, \overline{q_s u_s^{-1}})$$

in $\tilde{Y}$. Hence ξ is also injective. □

Lemma 3.2. *For any $u \in W^Q$, the map $Qu^{-1}B/B \to Q/B$ given by $\overline{qu^{-1}} \mapsto \overline{q}$ is well defined and Q-equivariant.*

Proof. Let $q_1, q_2 \in Q$ such that $q_1 u^{-1} B = q_2 u^{-1} B$. Then $u q_2^{-1} q_1 u^{-1} \in B$. It suffices to show that $q_2^{-1} q_1 \in B$. In other words, that $Q \cap u^{-1} B u \subseteq B$. We look at the set of roots corresponding to $Q \cap u^{-1} B u$. Since $u \in W^Q$, we have that $uR_Q^+ \subseteq R^+$ and $uR_Q^- \subseteq R^-$. Thus

$$R_Q \cap u^{-1} R^+ = u^{-1}(uR_Q \cap R^+) = u^{-1}(uR_Q^+) \subseteq R^+$$

and $Q \cap u^{-1} B u \subseteq B$. This proves the lemma. □

Assume we have $(\bar{g}\,;\,\bar{g}_1,\ldots,\bar{g}_s) \in G/P \times (G/B)^s$ such that $\bar{g} \in \bigcap_{k=1}^s g_k X_{w_k}$. It is easy to see that $(\bar{g}Q,\,;\,\bar{g}_1,\ldots,\bar{g}_s) \in Y$. By [2, Lemma 1], since $eP \in g^{-1}g_k X_{w_k}$, we have $g^{-1}g_k X_{w_k} = q_k u_k^{-1} X_{w_k}$ from some $q_k \in Q$. By Lemma 2.2,

$$g^{-1}g_k X_{w_k} \cap Q/P = q_k X_{v_k}.$$

We consider the points of Y that satisfy the following property.

Definition 3.3. *We say $(\bar{g}\,;\,\bar{g}_1,\ldots,\bar{g}_s) \in Y$ has property $P1$ if:*

(1) $\displaystyle\bigcap_{k=1}^s (g^{-1}g_k X_{w_k} \cap Q/P)$ *is transverse at every point in the intersection in Q/P*

(2) *For any $(q_1\ldots,q_s) \in Q^s$ such that, $g^{-1}g_k X_{u_k} = q_k u_k^{-1} X_{u_k} \subseteq G/Q$ for all k, the intersection*

$$\bigcap_{k=1}^s q_k X_{v_k} = \bigcap_{k=1}^s q_k \bar{X}_{v_k} \subseteq Q/P.$$

Proposition 3.4. *Property $P1$ is an open condition in Y.*

Proof. By Kleiman's transversality [6], there exists an open set $O \subseteq (Q/B)^s$ such that for any $(q_1,\ldots,q_s) \in O$ the following is satisfied:

(1) $\bigcap_{k=1}^s q_k X_{v_k} \subseteq Q/P$ is transverse at every point in the intersection.

(2) $\bigcap_{k=1}^s q_k X_{v_k} = \bigcap_{k=1}^s q_k \bar{X}_{v_k}$.

Moreover, we can choose O to be stable under the diagonal action of Q on $(Q/B)^s$ by replacing O with $\bigcup_{q\in Q} qO$. Consider the map

$$\tilde{\xi} : Y \to G \times_Q (Q/B)^s$$

defined by $\tilde{\xi} := \zeta \circ \xi^{-1}$ where

$$\zeta((g\,;\,\overline{q_1u_1^{-1}},\ldots,\overline{q_su_s^{-1}})) := (g\,;\,\overline{q_1},\ldots,\overline{q_s}).$$

By Lemma 3.2, the map $\tilde{\xi}$ is well defined and G-equivariant. Clearly any $(g\,;\,g_1,\ldots,g_s) \in \tilde{\xi}^{-1}(G \times_Q O)$ satisfies property $P1$. □

3.1. **Proof of Theorem 1.1.** Assume that $c_u \neq 0$. We first show there exists $(\bar{g}_1,\ldots\bar{g}_s) \in (G/B)^s$ which satisfies the following three conditions:

(1) $\bigcap_{k=1}^s g_k X_{w_k}$ is transverse at every point of the intersection in G/P and

$$\bigcap_{k=1}^s g_k X_{w_k} = \bigcap_{k=1}^s g_k \bar{X}_{w_k}.$$

(2) $\bigcap_{k=1}^s g_k X_{u_k}$ is transverse at every point of the intersection in G/Q and

$$\bigcap_{k=1}^s g_k X_{u_k} = \bigcap_{k=1}^s g_k \bar{X}_{u_k}.$$

(3) For every $x \in \bigcap_{k=1}^s g_k X_{u_k}$, we have that $(x\,;\,\bar{g}_1,\ldots,\bar{g}_s) \in Y$ has property $P1$.

By Kleimans tranversality [6], there exists an open $O_1 \subseteq (G/B)^s$ such that every point in O_1 satisfies conditions (1) and (2). By Proposition 3.4, there exists an open subset $Y^\circ \subseteq Y$ such that every point in Y° has property $P1$. Consider the projection of Y onto its second factor

$$\sigma : Y \to (G/B)^s.$$

Since $c_1 \neq 0$, the map σ is a dominant morphism. Moreover, the fibers of σ are generically finite and hence $\dim Y = \dim(G/B)^s$. Since Y is irreducible we have that

$$\dim \overline{\sigma(Y \backslash Y^\circ)} \leq \dim Y \backslash Y^\circ < \dim Y = \dim(G/B)^s.$$

Define the open set $O_2 := (G/B)^s \backslash (\overline{\sigma(Y \backslash Y^\circ)})$. Any $(\bar{g}_1, \ldots \bar{g}_s) \in O_1 \cap O_2$ satisfies conditions (1)-(3).

Assume that $(\bar{g}_1, \ldots, \bar{g}_s) \in O_1 \cap O_2 \subseteq (G/B)^s$. Conditions (1) and (2) imply that

$$\left| \bigcap_{k=1}^{s} g_k X_{w_k} \right| = c_w \text{ and } \left| \bigcap_{k=1}^{s} g_k X_{u_k} \right| = c_u.$$

Consider the G-equivariant projection $\pi : G/P \twoheadrightarrow G/Q$. If $\bar{g} \in \bigcap_{k=1}^{s} g_k X_{u_k}$, then $(\bar{g}\, ; \bar{g}_1, \ldots, \bar{g}_s) \in Y$ has property $P1$. Hence we have a surjection

$$\pi \left(\bigcap_{k=1}^{s} g_k X_{w_k} \right) = \bigcap_{k=1}^{s} g_k X_{u_k}.$$

Moreover, by Lemma 2.2, we have

$$\left| \bigcap_{k=1}^{s} g_k X_{w_k} \cap \pi^{-1}(\bar{g}) \right| = \left| \bigcap_{k=1}^{s} q_k u^{-1} X_{w_k} \cap Q/P \right| = \left| \bigcap_{k=1}^{s} q_k X_{v_k} \right| = c_v$$

where we choose $q_k \in Q$ such that $g^{-1} g_k X_{w_k} = q_k u_k^{-1} X_{w_k}$. Thus $c_w = c_u \cdot c_v$.

If $c_u = 0$, then $c_w = 0$ since for generic $(\bar{g}_1 \ldots, \bar{g}_s) \in (G/B)^s$, we have

$$\pi \left(\bigcap_{k=1}^{s} g_k X_{w_k} \right) \subseteq \bigcap_{k=1}^{s} g_k X_{u_k} = \emptyset.$$

Hence we still have $c_w = c_u \cdot c_v$. $\square$

4. Applications to Levi-movability

One application of the Theorem 1.1 is to compute structure coefficients corresponding to Levi-movable s-tuples in $(W^P)^s$. We begin with some preliminaries on Lie theory. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R^+$ be the set of simple roots of G where n is the rank of G. Note that the set Δ forms a basis for $\mathfrak{h}^*$ and let $\{x_1, x_2, \ldots, x_n\} \subseteq \mathfrak{h}$ be the dual basis to Δ such that

$$\alpha_i(x_j) = \delta_{i,j}.$$

Let $\Delta(P) \subset \Delta$ denote the simple roots associated to P (the simple roots that generate R_P^+). For any parabolic subgroup P and $w \in W^P$, define the character

$$\chi_w^P := \rho - 2\rho^P + w^{-1}\rho$$

where ρ is the half sum of all the roots in R^+ and ρ^P is the half sum of roots in R_P^+. The following proposition is proved in [2] using geometric invariant theory:

Proposition 4.1. *(Belkale and Kumar [2, Theorem 15]) If $(w_1, \ldots, w_s)$ is L_P-movable, then for every $\alpha_i \in \Delta \backslash \Delta(P)$, we have*

$$\big((\sum_{k=1}^{s} \chi_{w^k}^P) - \chi_1^P\big)(x_i) = 0.$$

4.1. **Proof of Theorem 1.3.** Recall that by Lemma 2.1, for any $w \in W^P$, we have $w = uv$ such that $u \in W^Q$ and $v \in W^P \cap W_Q$. For any pair of parabolic subgroups $P \subseteq Q$, let $T^P := T_{eP}(G/P)$ and $T^{P,Q} := T_{eP}(Q/P)$. For any $w \in W^P$ and $p \in P$ we have the subspace $pT_w^P := T_{eP}(pw^{-1}X_w) \subseteq T^P$. The condition for Levi-movability is equivalent to the condition that the diagonal map

$$\phi : T^P \to \bigoplus_{k=1}^{s} T^P/l_k T_{w_k}^P$$

is an isomorphism for generic $(l_1, \ldots, l_s) \in (L_P)^s$. Consider the diagram

$$\begin{tikzcd} T^{P,Q} \arrow[r, hook] \arrow[d, "\phi_2"] & T^P \arrow[r, two heads] \arrow[d, "\phi"] & T^Q \arrow[d, "\phi_1"] \\ \bigoplus_{k=1}^{s} \dfrac{T^{P,Q}}{l_k T_{v_k}^{P,Q}} \arrow[r, hook] & \bigoplus_{k=1}^{s} \dfrac{T^P}{l_k T_{w_k}^P} \arrow[r, two heads] & \bigoplus_{k=1}^{s} \dfrac{T^Q}{l_k v_k^{-1} T_{u_k}^Q} \end{tikzcd} \tag{6}$$

It suffices to show that if ϕ is an isomorphism, then ϕ_1 and ϕ_2 are isomorphisms.

Fix $(l_1, \ldots, l_s) \in L_P$ so that ϕ is an isomorphism. By the commutativity of the diagram (6), $\dim \operatorname{coker} \phi_1 = 0$, since $\dim \operatorname{coker} \phi = 0$. If we assume that $\dim \ker \phi_1 = 0$, then ϕ_1 is an isomorphism which proves part (1). Since ϕ is injective, ϕ_2 is also injective. By the snake lemma, we have that

$$\dim \ker \phi_1 = \dim \operatorname{coker} \phi_2 = 0.$$

Hence ϕ_2 is an isomorphism which proves part (2).

We now prove that $\dim \ker \phi_1 = 0$. Since ϕ_1 is surjective, the map

$$\phi_1 : T^Q/\ker \phi_1 \to \bigoplus_{k=1}^{s} \frac{T^Q}{l_i v_k^{-1} T_{u_k}^Q}$$

is an isomorphism. As a consequence, the induced map on top exterior powers:

$$\Phi_1 : \det(T^Q/\ker \phi_1) \to \det(\bigoplus_{k=1}^{s} \frac{T^Q}{l_k v_k^{-1} T_{u_k}^Q})$$

is nonzero. Identifying the character group $X(H)$ with the weight lattice in $\mathfrak{h}^*$ we have that $\mathfrak{h}$ acts on the complex line $\det(T^Q/\ker \phi_1)$ by the character $-\chi_1^Q - \beta$ where β is the

sum of roots in $\ker\phi_1$. Similarly, we have that $\mathfrak{h}$ acts diagonally on $\det(\bigoplus_{k=1}^{s}\frac{T^Q}{l_k v_k^{-1}T^Q_{u_k}})$ by the character $-\sum_{i=1}^{s}\chi^Q_{u_i}$. It is easy to see that the map Φ_1 is equivariant with respect to the action of the center of L_Q. In particular, for any $\alpha_i\in\Delta\backslash\Delta(Q)$, we have

$$(\chi^Q_1+\beta)(x_i)=\sum_{k=1}^{s}\chi^Q_{u_k}(x_i).$$

For any $w=uv\in W^P$ and $\alpha_i\in\Delta\backslash\Delta(Q)$, we have

$$\begin{aligned}\chi^P_w(x_i) &= (\rho-2\rho^P)(x_i)+w^{-1}\rho(x_i)\\ &= \rho(x_i)-\rho(uvx_i)\\ &= (\rho-2\rho^Q)(x_i)+u^{-1}\rho(x_i)\\ &= \chi^Q_u(x_i)\end{aligned}$$

since the Weyl group W_Q acts trivially on x_i and $\rho^P(x_i)=\rho^Q(x_i)=0$. Hence, by Proposition 4.1, we have

$$\beta(x_i)=\big((\sum_{k=1}^{s}\chi^Q_{u_i})-\chi^Q_1\big)(x_i)=\big((\sum_{i=1}^{s}\chi^P_{w_i})-\chi^P_1\big)(x_i)=0$$

for all $\alpha_i\in\Delta\backslash\Delta(Q)$. But

$$\ker\phi_1\subseteq T^Q=\bigoplus_{\alpha\in R^-\backslash R^-_Q}\mathfrak{g}_\alpha$$

and hence $-\beta$ is a positive linear combination of positive simple roots in $\Delta\backslash\Delta(Q)$. Thus $\ker\phi_1=0$. This proves Theorem 1.3. □

5. Branching Schubert calculus

In this section we generalize Theorems 1.1 and 1.3 to the setting of branching Schubert calculus. These generalizations are stated in Theorems 1.6 and 1.8. Since the proofs are similar to those for Theorems 1.1 and 1.3, we leave several details to the reader. Let $\tilde{G}$ be any connected semisimple subgroup of G and fix a maximal torus $\tilde{H}\subseteq\tilde{G}$ such that $\tilde{H}=H\cap\tilde{G}$. Fix a Borel subgroup $\tilde{B}\subseteq\tilde{G}$ which contains $\tilde{H}$ (we do not assume that $\tilde{B}\subseteq B$) and fix $z\in N_G(H)$ such that

$$\tilde{B}=zBz^{-1}\cap\tilde{G}.$$

By [3, Proposition 2.2.6], their always exists such a $z\in N_G(H)$ however this choice is not unique. As in Theorem 1.1, we consider a pair of parabolic subgroups $P\subseteq Q$ in G which contain B. Define parabolic subgroups

$$\tilde{P}:=zPz^{-1}\cap\tilde{G}$$
$$\tilde{Q}:=zQz^{-1}\cap\tilde{G}$$

and consider the maps

$$\phi : \tilde{G}/\tilde{P} \hookrightarrow G/P$$
$$\phi_1 : \tilde{G}/\tilde{Q} \hookrightarrow G/Q$$
$$\phi_2 : \tilde{Q}/\tilde{P} \to Q/P$$

defined by $\phi(g\tilde{P}) := z^{-1}gzP$, $\phi_1(g\tilde{Q}) := z^{-1}gzQ$ and $\phi_2 := \phi|_{\tilde{Q}/\tilde{P}}$. Consider the following commuting diagram:

$$\begin{tikzcd} \tilde{Q}/\tilde{P} \arrow[r, hook] \arrow[d, "\phi_2"] & \tilde{G}/\tilde{P} \arrow[r, two heads, "\pi"] \arrow[d, "\phi"] & \tilde{G}/\tilde{Q} \arrow[d, "\phi_1"] \\ Q/P \arrow[r, hook] & G/P \arrow[r, two heads] & G/Q \end{tikzcd} \tag{7}$$

For any $w \in W^P$ such that $\dim X_w = \dim G/P - \dim \tilde{G}/\tilde{P}$, we have the associated structure constant $c_w \in \mathbb{Z}_{\geq 0}$ defined by

$$\phi^*([X_w]) = c_w[pt].$$

By Lemma 2.1, we can write $w = uv$ where $u \in W^Q$ and $v \in W^P \cap W_Q$. We show that if that condition (4) is satisfied, then $c_w = c_u \cdot c_v$ where

$$\phi_1^*([X_u]) = c_u[pt]$$
$$\phi_2^*([X_v]) = c_v[pt].$$

5.1. **Proof of Theorem 1.6.** If $w \in W^P$ satisfies condition (4), then there exists an open subset $O_1 \subseteq G/B$, such that if $\bar{g} \in O_1$, then the cardinality of inverse images

$$|\phi^{-1}(gX_w)| = c_w \text{ and } |\phi_1^{-1}(gX_u)| = c_u.$$

Consider the projection $\pi : \tilde{G}/\tilde{P} \to \tilde{G}/\tilde{Q}$. By the commutativity of diagram (7), we have that $\pi(\phi^{-1}(gX_w)) \subseteq \phi_1^{-1}(gX_u)$. Hence if $c_u = 0$, then $c_w = 0$. Assume that $c_u \neq 0$.

It suffices to show that for generic $\bar{g} \in G/B$, the map π restricted to $\phi^{-1}(gX_w)$ is surjective and for any $\bar{h} \in \phi_1^{-1}(gX_u)$, we have $|\pi^{-1}(\bar{h}) \cap \phi^{-1}(gX_w)| = c_v$.

Following the proof of Theorem 1.1, we define the set

$$Y := \{(\bar{h}, \bar{g}) \in \tilde{G}/\tilde{Q} \times G/B \mid \phi(\bar{h}) \in gX_u\}.$$

By analogues of Lemma 3.1 and Proposition 3.4, the set Y is a smooth irreducible $\tilde{G}$-variety and the following property $P2$ is an open condition on Y:

Definition 5.1. *We say $(\bar{h}, \bar{g}) \in Y$ has property $P2$ if:*

(1) *The intersection $(z^{-1}h^{-1}zgX_w \cap Q/P) \cap \phi_2(\tilde{Q}/\tilde{P})$ is transverse at every point in Q/P.*
(2) *For any $q \in Q$ such that $z^{-1}h^{-1}zgX_u = qu^{-1}X_u \subseteq G/Q$, the intersection*

$$qX_v \cap \phi_2(\tilde{Q}/\tilde{P}) = q\bar{X}_v \cap \phi_2(\tilde{Q}/\tilde{P}) \subseteq Q/P.$$

Let $Y^\circ \subseteq Y$ be an open set whose points have property $P2$ and let $\sigma : Y \to G/B$ denote the projection onto the second factor of Y. By the proof of Theorem 1.1, the set $O_2 := G/B \backslash \overline{\sigma(Y \backslash Y^\circ)}$ is an open subset of G/B.

Moveover, if $g \in O_1 \cap O_2$, then $\pi(\phi^{-1}(gX_w)) = \phi_1^{-1}(gX_u)$. By [2, Lemma 1], we can choose $q \in Q$ such that $z^{-1}h^{-1}zgX_w = qu^{-1}X_w$. By Lemma 2.2, for any $\bar{h} \in \phi_1^{-1}(gX_u)$, we have

$$|\pi^{-1}(\bar{h}) \cap \phi^{-1}(gX_w)| = |qu^{-1}X_w \cap Q/P \cap \phi_2(\tilde{Q}/\tilde{P})| = |qX_v \cap \phi_2(\tilde{Q}/\tilde{P})| = c_v.$$

Hence $c_w = c_u \cdot c_v$. This proves Theorem 1.6. □

5.2. **Proof of Theorem 1.8.** Let $\tilde{R}$ denote the set of roots of $\tilde{G}$ with respect to the torus $\tilde{H}$ and let $\tilde{R}^+$ denote the set of positive roots with respect to the Borel $\tilde{B}$. Let $\Delta_{\tilde{G}} := \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\} \subseteq \tilde{R}^+$ denote the of simple roots of $\tilde{G}$ where m is the rank of $\tilde{G}$. Let $\{\tilde{x}_1, \ldots, \tilde{x}_m\} \subseteq \tilde{\mathfrak{h}}$ denote the dual basis to $\Delta_{\tilde{G}}$. For any parabolic subgroup $\tilde{Q} \subseteq \tilde{G}$ which contains $\tilde{B}$, let $\tilde{R}^+_{\tilde{Q}}$ denote the positive roots of of $\tilde{Q}$ or $L_{\tilde{Q}}$ and let $\Delta_{\tilde{Q}} := \Delta_{\tilde{G}}(\tilde{Q}) \subseteq \Delta_{\tilde{G}}$ denote the corresponding simples roots. Consider the following diagram which is analogous to (6). By an abuse of notation we will use ϕ, ϕ_1, ϕ_2 to denote the induced map on Lie algebras.

(8)
$$\begin{tikzcd}
\tilde{T}^{P,Q} \arrow[r, hook] \arrow[d, "\phi_2"] & \tilde{T}^{P} \arrow[r, two heads] \arrow[d, "\phi"] & \tilde{T}^{Q} \arrow[d, "\phi_1"] \\
\dfrac{T^{P,Q}}{lT_v^{P,Q}} \arrow[r, hook] & \dfrac{T^{P}}{lT_w^{P}} \arrow[r, two heads] & \dfrac{T^{Q}}{lv^{-1}T_u^{Q}}
\end{tikzcd}$$

Since $w \in W^P$ is (L_P, ϕ)-movable, the map ϕ is an isomorphism. By the snake lemma, it suffices to show that ϕ_1 is injective. Let $\beta \in \tilde{\mathfrak{h}}^*$ denote the sum of roots corresponding to $\ker \phi_1$. Following the proof of Theorem 1.3, it suffices to show that $\beta(\tilde{x}_i) = 0$ for all $\tilde{\alpha}_i \in \Delta_{\tilde{G}} \backslash \Delta_{\tilde{Q}}$ since $\ker \phi_1 \subseteq \tilde{T}^Q$.

Consider the group

$$C := \tilde{H} \cap zZ(L_Q)z^{-1}$$

where $Z(L_Q)$ denotes the center of L_Q. Observe that $C \subseteq Z(L_{\tilde{Q}})$ and that $\mathrm{Lie}(C) = \tilde{\mathfrak{h}} \cap z\mathfrak{z}z^{-1}$ where $\mathfrak{z}$ denotes the Lie algebra of $Z(L_Q)$. Since $C \subseteq \tilde{H}$, we have induced C-module structures on $\tilde{T}^P, \tilde{T}^Q, \tilde{T}^{P,Q}$. Similarly, since $C \subseteq zZ(L_Q)z^{-1}$, we have induced twisted C-module structures on $T^P, T^Q, T^{P,Q}$ given by the action

$$g \odot t := z^{-1}gzt.$$

It is easy to see that the maps ϕ, ϕ_1 and ϕ_2 are C-equivariant with respect to these C-actions. Since ϕ is an isomorphism and ϕ_1 is surjective, the induced C-equivariant maps

$$\Phi : \det(\tilde{T}^P) \to \det(T^P/lT_w^P)$$

and

$$\Phi_1 : \det(\tilde{T}^Q/\ker \phi_1) \to \det(T^Q/lv^{-1}T_u^Q)$$

are nonzero.

Define the map $i_z : \tilde{G} \hookrightarrow G$ by $i_z(g) := z^{-1}gz \in G$ and the character

$$\tilde{\chi}^{\tilde{P}} := 2(\tilde{\rho} - \tilde{\rho}^{\tilde{P}})$$

where $\tilde{\rho}$ is the half sum of all roots in $\tilde{R}^+$ and $\tilde{\rho}^{\tilde{P}}$ is the half sum of all roots in $\tilde{R}^+_{\tilde{P}}$. Observe that for any $\tau \in \mathrm{Lie}(C)$ we have

$$\beta(\tau) = \left(i_z^*\chi_u^Q - \tilde{\chi}^{\tilde{Q}}\right)(\tau) = \left(i_z^*\chi_w^P - \tilde{\chi}^{\tilde{P}}\right)(\tau) = 0$$

since the isomorphisms Φ and Φ_1 are C-equivariant. By assumptions in Theorem 1.8, there exists a vector $\tau_0 \in \mathrm{Lie}(C)$ such that $\alpha(\tau_0) \geq 0$ for any $\alpha \in \Delta_{\tilde{G}}$ with equality if and only if $\alpha \in \Delta_{\tilde{Q}}$. This implies that $\beta(\tilde{x}_i) = 0 \quad \forall\, \tilde{\alpha}_i \in \Delta_{\tilde{G}} \backslash \Delta_{\tilde{Q}}$. This proves Theorem 1.8. □

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